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Controlled synchronization under information constraints

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A class of controlled synchronization systems under information constraints imposed by limited information capacity of the coupling channel is analyzed. It is shown that the framework proposed by Fradkov *et al.*, [Phys. Rev. E **73**, 066209 (2006)] is suitable not only for observer-based synchronization but also for controlled master-slave synchronization via a communication channel with limited information capacity. A simple first-order coder-decoder scheme is proposed and a theoretical analysis for multidimensional master-slave systems represented in the Lurie form (linear part plus nonlinearity depending only on measurable outputs) is provided. An output feedback control law is proposed based on the passification method. It is shown that for systems with passifiable linear part (satisfying the hyperminimum phase condition) the upper bound of the limiting synchronization error is proportional to the upper bound of the transmission error. As a consequence, both upper and lower bounds of the limiting synchronization error are proportional to the maximum rate of the coupling signal and inversely proportional to the information transmission rate (channel capacity). The results are applied to controlled synchronization of two chaotic Chua systems coupled via a controller and a channel with limited capacity. It is shown by computer simulation that, unlike for the case of observer-based synchronization, the hyperminimum phase property cannot be violated for controlled synchronization.

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I. INTRODUCTION

Synchronization of nonlinear systems, particularly chaotic systems, has attracted the attention of many researchers for several decades [1,2]. During recent years interest in controlled synchronization has increased, partly driven by a growing interest in the application of control theory methods in physics [3–6].

Recently, the limitations of control under constraints imposed by a finite capacity information channel have been investigated in detail in the control theory literature (see [7–11] and the references therein). It has been shown that stabilization under information constraints is possible if and only if the capacity of the information channel exceeds the entropy production of the system at equilibrium. However, results of previous work on control system analysis under information constraints do not apply to synchronization systems since in a synchronization problem the trajectories in phase space converge to a set (a manifold) rather than to a point, i.e., the problem cannot be reduced to simple stabilization.

The first results on synchronization under information constraints were presented in [12], where the so-called observer-based synchronization scheme [13] was considered. In this paper we study a different controlled synchronization scheme for two nonlinear systems. A major difficulty with the controlled synchronization problem arises because the coupling is implemented in a restricted manner via the control signal. An arbitrary change of the right-hand sides of the receiver is not possible; only changes via an additional scalar control signal are admitted. Such structural restrictions are important for physically implemented real-world systems. In addition, the slave system in a controlled synchronization scheme is nonlinear and bears the structure of the master system, unlike the observer-based synchronization case.

In this paper, we provide structural conditions [the *hyperminimum phase* (HMP) property] ensuring that increase of the transmission rate to infinity implies decrease of the synchronization error to zero (see Sec. III). Moreover, it is shown in Sec. IV that violation of the HMP property may lead to synchronization failure even if the conditions on the linear part, imposed in [12], hold. Key tools used to solve the problem are quadratic Lyapunov functions and passification methods borrowed from control theory. To reduce technicalities we restrict our analysis to Lurie systems (linear part plus nonlinearity depending only on measurable outputs). In this case the physical insight behind the HMP property is that this property makes a system *passifiable* by feedback. In turn, passivity of a dynamical system means that the system does not contain internal energy sources. Recall that the design method based on finding a feedback rendering the system passive is called *the passification method*.

II. CONTROLLED SYNCHRONIZATION SCHEME AND CODING PROCEDURES

Consider two identical dynamical systems modeled in Lurie form (i.e., the right-hand sides are split into a linear part

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and a nonlinear part which depends only on the measurable outputs). Let one of the systems be controlled by a scalar control function $u(t)$ implementing the coupling between two systems. The controlled system model is as follows:

$$\dot{x}(t) = Ax(t) + B\varphi(y_1), \quad y_1(t) = Cx(t), \tag{1}$$

$$\dot{z}(t) = Az(t) + B\varphi(y_2) + Du, \quad y_2(t) = Cz(t), \tag{2}$$

where $x(t)$ and $z(t)$ are n-dimensional (column) vectors of state variables; $y_1(t)$, $y_2(t)$ are scalar output variables; A is an $(n\times n)$ matrix; B and D are $n\times 1$ (column) matrices (D is a matrix of coupling strengths); C is a $1\times n$ (row) matrix; $\varphi(y)$ is a continuous nonlinearity, acting in the span of control; vectors $\dot{x}$ and $\dot{z}$ stand for time derivatives of $x(t)$ and $z(t)$, respectively. System (1) is called the *master* (*leader*) *system*, while the controlled system (2) is the *slave system* (*follower*). Our goal is to evaluate limitations imposed on the synchronization precision by limiting the transmission rate between the systems. The intermediate problem is to find a control function $\mathcal{U}(\cdot)$ depending on measurable variables such that the synchronization error $e(t)$, where $e(t)=z(t)-x(t)$, becomes small as t becomes large. We are also interested in the value of the output synchronization error $\varepsilon(t)=y_2(t)-y_1(t)=Ce(t)$.

A key difficulty arises because the output of the master system is not available directly but only through a communication channel with limited capacity. This means that the signal $y_1(t)$ must be coded at the transmitter side and codewords are then transmitted with only a finite number of symbols per second, thus introducing some error. We assume that the observed signal $y_1(t)$ is coded with symbols from a finite alphabet at discrete sampling time instants $t_k=kT_s$, $k=0,1,2,\ldots$, where T_s is the sampling time. Let the coded symbol $\bar{y}_1[k]=\bar{y}_1(t_k)$ be transmitted over a digital communication channel with a finite capacity. To simplify the analysis, we assume that the observations are not corrupted by observation noise; transmission delay and transmission channel distortions may be neglected. Therefore, the discrete communication channel with sampling period T_s is considered, but it is assumed that the coded symbols are available at the receiver side at the same sampling instant $t_k=kT_s$, as they are generated by the coder. Assume that *zero-order extrapolation* is used to convert the digital sequence $\bar{y}_1[k]$ to the continuous-time input of the response system $\bar{y}_1(t)$, namely, that $\bar{y}_1(t)=\bar{y}_1[k]$ as $kT_s \leqslant t<(k+1)T_s$. Then the *transmission error* is defined as follows:

$$\delta_y(t) = y_1(t) - \bar{y}_1(t). \tag{3}$$

On the receiver side the signal is decoded, introducing additional error, and the controller can use only the signal $\bar{y}_1(t)=y_1(t)+\delta_y(t)$ instead of $y(t)$. A block diagram of the system is shown in Fig. 1.

We restrict consideration to simple control functions in the form of static linear feedback,

$$u(t) = -K\varepsilon(t), \tag{4}$$

where $\varepsilon(t)=y_2(t)-y_1(t)$ denotes an output synchronization error, and K is a scalar controller gain. The problem of finding static output feedback even for linear systems is one of the classical problems of control theory. Although substantial effort has been devoted to its solution and various necessary and sufficient conditions for stabilizability by static output feedback have been obtained, most existing conditions are not testable practically [14,15]. Since we are dealing with a nonlinear problem further complicated by information constraints, we restrict our attention to sufficient conditions for solvability of the problem and evaluate upper bounds for the synchronization error. To this end we introduce an upper bound on the limit synchronization error $Q=\sup \overline{\lim}_{t\to\infty}\|e(t)\|$, where the supremum is taken over all admissible transmission errors.

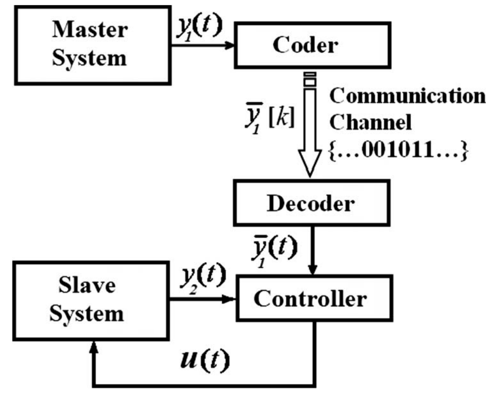

FIG. 1. Block diagram for master-slave controlled synchronization (the master system output y_1 is transmitted over the channel).

In [12] the properties of observer-based synchronization for Lurie systems over a limited-band communication channel with one-step memory time-varying coder are studied. Under the assumption that a sampling period may be properly chosen, optimality of the binary coding in the sense of the demanded transmission rate is established. On the basis of these results, the present paper deals with the following binary coding procedure as in [12].

Introduce the memoryless (static) binary coder to be a discretized map $q_M:\mathbb{R}\to\mathbb{R}$ as

$$q_M(y) = M\,\mathrm{sgn}(y), \tag{5}$$

where $\mathrm{sgn}(\cdot)$ is the *signum* function: $\mathrm{sgn}(y)=1$ if $y\geqslant 0$, $\mathrm{sgn}(y)=-1$ if $y<0$; the parameter M may be referred to as a *coder range* or as a *saturation value*. Evidently, $|y-q_M(y)|\leqslant M$ for all y such that $y:|y|\leqslant 2M$. Notice that for a binary coder each codeword symbol contains $R=1$ bit. The discretized output of the considered coder is found as $\bar{y}=q_M(y)$ and we assume that the coder and decoder make decisions based on the same information [16–18].

Introduce the sequence of *central numbers* $c[k]$, $k=0,1,2,\ldots$, with initial condition $c[0]=0$. At step k the coder compares the current measured output $y[k]$ with the number $c[k]$, forming the deviation signal $\partial y[k]=y[k]-c[k]$. Then this signal is discretized with a given $M=M[k]$ according to (5). The output signal

$$\bar{\partial} y[k] = q_{M[k]}(\partial y[k]) \tag{6}$$

is represented as an R-bit information symbol from the coding alphabet and transmitted over the communication channel to the decoder. Then the central number $c[k+1]$ and the range parameter $M[k]$ are renewed as follows:

$$c[k+1] = c[k] + \bar{\partial}y[k], \quad c[0] = 0, \quad k = 0,1, \ldots, \tag{7}$$

$$M[k] = (M_0 - M_\infty)\rho^k + M_\infty, \quad k = 0,1, \ldots, \tag{8}$$

where $0 < \rho \leqslant 1$ is the decay parameter, and M_∞ stands for the limiting value of $M[k]$. The initial value M_0 should be large enough to capture all the region of possible initial values of y_0.

A similar algorithm is realized by the decoder; namely, the sequence of $M[k]$ is reproduced at the receiver node utilizing (8); the values of $\bar{\partial}y[k]$ are restored with given $M[k]$ from the received codeword; the central numbers $c[k]$ are found in the decoder in accordance with (7). Then $\bar{y}[k]$ is found as a sum $c[k]+\bar{\partial}y[k]$.

III. EVALUATION OF SYNCHRONIZATION ERROR

We now find a relation between the transmission rate and the achievable accuracy of the coder-decoder pair, assuming that the growth rate of $y_1(t)$ is uniformly bounded. Obviously, the exact bound L_y for the rate of $y(t)$ is $L_y = \sup_{x\in\Omega}|C\dot{x}|$, where $\dot{x}$ is from (2). To analyze the coder-decoder accuracy, evaluate the upper bound $\Delta = \sup_t|\delta_y(t)|$ of the transmission error $\delta_y(t) = y_1(t) - \bar{y}_1(t)$. The total transmission error for each interval $[t_k, t_{k+1}]$ satisfies the inequality

$$|\delta_y(t)| \leqslant M + L_y T_s. \tag{9}$$

The inequality (9) shows that, in order to meet the inequality $|\delta_y(t)| \leqslant \Delta = 2M$ for all t, the sampling interval T_s should satisfy the condition

$$T_s < \Delta / L_y. \tag{10}$$

To evaluate the limit synchronization error analytically, it is assumed that $D=B$ in (2). It is also assumed that the transfer function $W(\lambda) = C(\lambda I - A)^{-1}B$ of the linear part of the system (2) is a hyperminimum phase. Recall that the HMP property for a rational function $W(\lambda) = b(\lambda)/a(\lambda)$, where $a(\lambda)$ is a polynomial of degree n and $b(\lambda)$ is a polynomial of degree not greater than $n-1$, means that $b(\lambda)$ is a Hurwitz polynomial of degree $n-1$ with positive coefficients [19].

Evaluation of the synchronization error (see Appendix A) yields

$$\overline{\lim_{t\to\infty}}\|e(t)\| \leqslant C_e^+ \beta L_y / R, \tag{11}$$

i.e., this error can be made arbitrarily small for sufficiently large transmission rate R.

Remark 1. The HMP property is essential for validity of the proposed solution to the controlled synchronization problem, as is shown below in Sec. IV by example of chaotic system synchronization. On the other hand, the condition $D=B$, used for the analytic derivation, is not essential.

Remark 2. Related estimates for synchronization errors in coupled systems were obtained in several papers, e.g., [20,21]. However, in [20,21] either the existence of Lyapunov functions, i.e., stability of the uncoupled systems is required, or partial stability (stability of the synchronization manifold) is provided by a strong coupling playing the role of state feedback in the error system. In this paper, only output feedback is allowed, and coupling is applied through the control term Bu, i.e., in a restrictive manner. That is why the result holds under the additional assumption (passifiability) caused by the nature of controlled problems. Then the partial stability conditions are provided by linear observer theory. In addition the final result (11) is presented in terms of the transmission rate, i.e., appeals to the information theory view.

IV. EXAMPLE: SYNCHRONIZATION OF CHAOTIC CHUA SYSTEMS

Let us apply the above results to synchronization of two chaotic Chua systems coupled via a channel with limited capacity.

Master system. Let the master system (1) be represented as the following Chua system:

$$\dot{x}_1 = p(-x_1 + \varphi(y_1) + x_2), \quad t \geqslant 0,$$

$$\dot{x}_2 = x_1 - x_2 + x_3,$$

$$\dot{x}_3 = -qx_2,$$

$$y_1(t) = x_1(t), \tag{12}$$

where $y_1(t)$ is the master system output (to be transmitted over the communication channel); p and q are known parameters; $x=[x_1,x_2,x_3]^T \in \mathbb{R}^3$ is the state vector; $\varphi(y_1)$ is a piecewise-linear function, having the form

$$\varphi(y) = m_0 y + m_1(|y+1| - |y-1|), \tag{13}$$

where m_0 and m_1 are given parameters.

Slave system. Correspondingly, the slave system equations (1) for the considered case become

$$\dot{z}_1 = p[-z_1 + \varphi(y_2) + z_2 + u(t)], \quad t \geqslant 0,$$

$$\dot{z}_2 = z_1 - z_2 + z_3,$$

$$\dot{z}_3 = -qx_2,$$

$$y_2(t) = z_1(t), \tag{14}$$

where $y_2(t)$ is the slave system output, $z=[z_1,z_2,z_3]^T \in \mathbb{R}^3$ is the state vector, and $\varphi(y_2)$ is defined by (13).

The *controller* has the form

$$u(t) = -K\varepsilon(t), \tag{15}$$

swhere $\varepsilon(t) = y_2(t) - \bar{y}_1(t)$; $\bar{y}_1(t)$ is the master system output, restored from the transmitted codeword by the receiver at the slave system node (see Fig. 1); the gain K is a design parameter.

The *coding procedure* has the form (6)–(8). The input signal of the coder is $y_1(t)$. The reference input $\bar{y}_1(t)$ for controller (15) is found by holding the value of $\bar{y}_1[k]$ over the sampling interval $[kT_s,(k+1)T_s)$, $k=0,1,\ldots$.

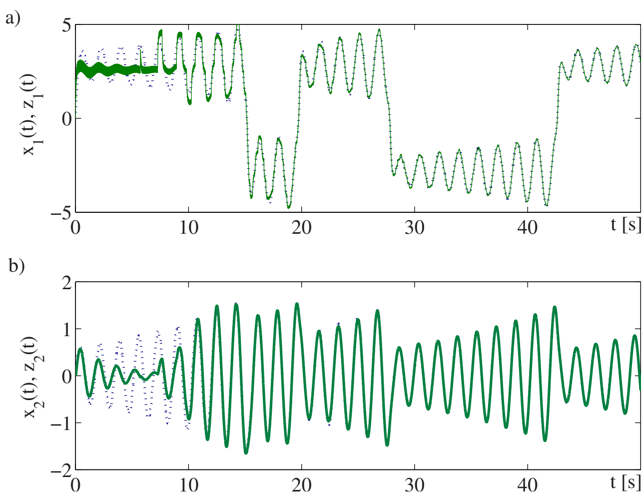

FIG. 2. (Color online) Time histories of the state variables of master and slave systems (12) and (14) for $\Delta=1$ ($T_s=13$ ms, $R=75$ bit/s): (a) $x_1(t)$ (dotted line, $z_1(t)$ (solid line); (b) $x_2(t)$ (dotted line, $z_2(t)$ (solid line).

The following parameter values were taken for the simulation: the Chua system parameters are $p=10$, $q=15.6$, $m_0=0.33$, $m_1=0.945$; the bound L_y for the rate of $y_1(t)$ was evaluated by numeric integration of (1) over the time interval $t\in[0,t_{\rm fin}]$, $t_{\rm fin}=1000$ s, as $L_y=45$; the parameter Δ was taken for different simulation runs as $\Delta=0.2,0.4,\ldots,3.0$; the sample interval T_s was found for each Δ from (10); the coder parameters M_0, M_∞, and ρ in (8) were taken as $M_0=5$, $M_\infty=\Delta/2$ (different for each Δ), and $\rho=\exp(-0.1T_s)$; the initial conditions for master and slave systems were $x_i=0.3$, $z_i=0$ ($i=1,2,3$); and the simulation final time $t_{\rm fin}=1000$ s.

The *normalized state synchronization error*

$$Q=\frac{\max\limits_{0.8t_{\rm fin}\leq t\leq t_{\rm fin}}\|e(t)\|}{\max\limits_{0\leq t\leq t_{\rm fin}}\|x(t)\|}, \tag{16}$$

where $\delta_y(t)=y_1(t)-\bar{y}_1(t)$, $e(t)=x(t)-z(t)$, was calculated by performing the synchonization accuracy index near a steady-state mode.

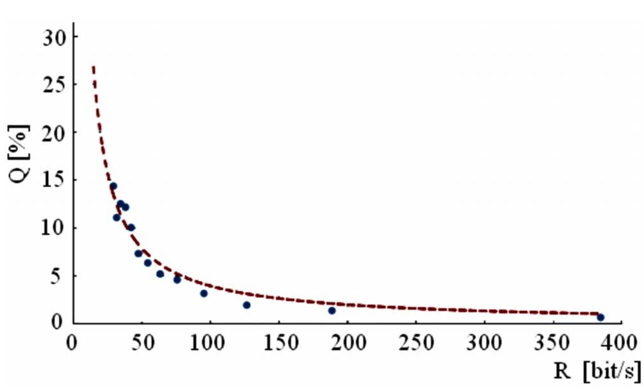

FIG. 3. (Color online) Normalized state synchronization error Q vs transmission rate R.

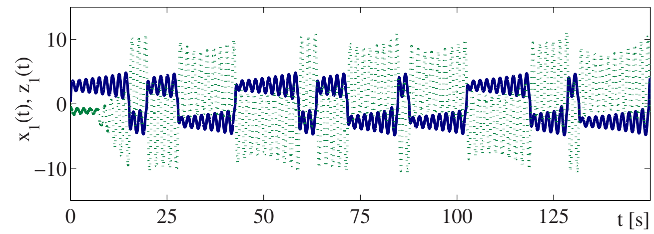

FIG. 4. (Color online) Time histories of the state variables of master and slave systems (12) and (14): $x_1(t)$ (solid) line, $z_1(t)$ (dotted line) ($T_s=13$ ms, $R=75$ bit/s). Synchronization failure if HMP property is not valid for (2).

Results of the system examination for $K=1.0$ are reflected in Figs. 2 and 3. Figure 2 shows time histories of the state variables of the master and slave systems (12) and (14) $x_1(t),z_1(t)$ (a) and $x_2(t),z_2(t)$ (b) for $\Delta=1$ ($T_s=13$ ms, $R=75$ bit/s). It is seen that the transient time is about 30 s, which is consistent with the chosen value of the decay parameter ρ in (8), $\rho=\exp(-0.1T_s)=0.9987$.

Synchronization performance may be evaluated based on the normalized state synchronization error Q (16), shown in Fig. 3 as a function of the transmission rate R. The simulation results make it possible to evaluate the parameter G_y in the inversely proportional function $Q=G/R$. For the considered example $G=4.0$. Based on the theoretical bound, the simulation data are smothed with a hyperbolic curve, plotted in Fig. 3 (dashed line).

Let us demonstrate that violation of the HMP condition may lead to synchronization failure. Indeed, let $D=[-4.66,0.5,-4.4]^T$. Then synchronization fails for all values of K. It is seen from the time histories of $x_1(t)$ and $z_1(t)$ for the case $K=1$ plotted in Fig. 4. On the other hand, for the case $D=[1.00,5.54,4.44]^T$ the HMP condition holds while $D\neq B$. Simulation results for $D=[1.00,5.54,4.44]^T$, $K=20$, demonstrate that synchronization occurs (see Fig. 5).

V. CONCLUSIONS

The limiting possibilities of controlled synchronization systems under information constraints imposed by the limited information capacity of the coupling channel are evaluated. It is shown that the framework proposed in [12] is suitable not only for observer-based synchronization but also for controlled master-slave synchronization via communica-

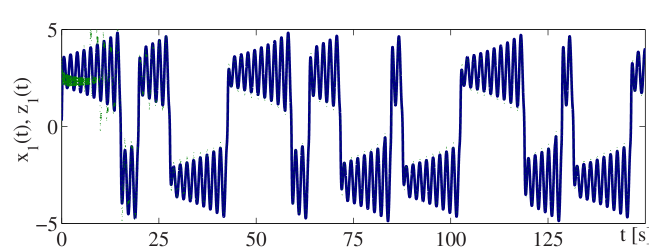

FIG. 5. (Color online) Time histories of the state variables of master and slave systems (12) and (14): $x_1(t)$ (solid) line, $z_1(t)$ (dotted line) ($T_s=13$ ms, $R=75$ bit/s). Synchronization occurs if $D\neq B$; the HMP property is valid for (2).

tion channel with limited information capacity.

The output feedback control law based on the passification method [19,22] is proposed, and theoretical analysis for multidimensional master-slave systems represented in the Lurie form is provided. It is shown that the upper bound on the limiting synchronization error is proportional to the maximum rate of the coupling signal and inversely proportional to the information transmission rate (channel capacity).

The results are applied to controlled synchronization of two chaotic Chua systems coupled via a channel with limited capacity. It is shown by computer simulation that, unlike the case of observer-based synchronization, the HMP property cannot be violated for controlled synchronization.

ACKNOWLEDGMENTS

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APPENDIX A: DERIVATION OF THE UPPER BOUND OF SYNCHRONIZATION ERROR

Subtracting Eq. (1) from Eq. (2) and taking into account the control law (4), we derive an equation for the synchronization error in the form

$$\dot{e}(t) = A_K e(t) + B\zeta(t) - BK\delta_y(t), \quad \text{(A1)}$$

where $A_K = A - BKC$ and $\zeta(t) = \varphi(y_2(t)) - \varphi(y_1(t))$. We evaluate the total guaranteed synchronization error $Q = \sup \overline{\lim}_{t\to\infty}\|e(t)\|$, where $\|\cdot\|$ denotes the Euclidean norm of a vector, and the supremum is taken over all admissible transmission errors $\delta_y(t)$ not exceeding the level Δ in absolute value. The ratio $C_e = Q/\Delta$ (the relative error) can be interpreted as the norm of the transformation from the input function $\delta_y(\cdot)$ to the output function $e(\cdot)$ generated by the system (A1). Owing to the nonlinearity of Eq. (A1), evaluation of the norm C_e is nontrivial and it even may be infinite for rapidly growing nonlinearities $\varphi(y)$. To obtain a reasonable upper bound for C_e we assume that the nonlinearity is Lipschitz continuous along all the trajectories of the drive system (2). More precisely, we assume the existence of a positive number $L_\varphi > 0$ such that

$$|\varphi(y) - \varphi(y+\delta)| \leq L_\varphi |\delta|$$

for all $y = Cx$, $x \in \Omega$, where Ω is a set containing all the trajectories of the drive system (1), starting from the set of initial conditions Ω_0, $|\delta| \leq \Delta$. For Lipschitz nonlinearities $\zeta(t)$ satisfies inequality $|\zeta(t)| \leq L_\varphi |\varepsilon(t)|$. After the change $K \to K + L_\varphi$, the error equation (A1) can be represented as

$$\dot{e}(t) = A_K e(t) + B\xi(t) - B(K + L_\varphi)\delta_y(t), \quad \text{(A2)}$$

where the variable $\xi(t) = L_\varphi \varepsilon(t) + \zeta$, apparently, satisfies sector inequality $\xi(t)\varepsilon(t) \geq 0$ for all $t \geq 0$.

The problem is reduced to quantifying the stability properties of (A2) for bounded input $\delta_y(t)$. We first analyze the behavior of the system (A2) for $\delta_y(t) = 0$. To this end, we find conditions for the existence of a quadratic Lyapunov function $V(e) = e^T P e$ and controller gain K satisfying the inequality $\dot{V}(e) \leq -\mu V(e)$ for some $\mu > 0$ for $\delta_y(t) = 0$ and for all ξ satisfying the quadratic inequality $\xi\varepsilon \geq 0$. Such conditions are derived from the passification theorem [19,22] (see Appendix B). That is, such V and K exist if and only if the transfer function of the linear part of (2) $W(\lambda) = C(\lambda I - A)^{-1}B$ is the hyperminimum phase. Now consider the case $\delta_y(t) \neq 0$, assuming that the HMP condition holds and the matrix P and gain K are chosen properly and the modified Lyapunov inequality $PA_K + A_K^T P \leq -\mu P$ is valid for some $\mu > 0$. Evaluating the time derivative of function $V(e)$ along trajectories of (2) and (1) with initial conditions in Ω_0 and using the standard quadratic inequality $|e^T PB| \leq \sqrt{V(e)}\sqrt{V(B)}$, after simple algebra, we get

$$\dot{V} \leq -\mu V + |e^T PB(K + L_\varphi)\delta_y| \leq -\mu V + \sqrt{V}\nu,$$

where $\nu = \sqrt{V(B)}(|K| + L_\varphi)\Delta$. Since $\dot{V} < 0$ within the set $\sqrt{V} > \mu^{-1}\nu$, the value of $\lim_{t\to\infty} \sup V(t)$ cannot exceed $\Delta^2(L_\varphi + |K|)^2 \lambda_{\max}(P)/\mu^2$. In view of the positivity of P, $\lambda_{\min}(P)\|e(t)\|^2 \leq V(t)$, where $\lambda_{\min}(P)$ and $\lambda_{\max}(P)$ are the minimum and maximum eigenvalues of P, respectively. Hence

$$\overline{\lim_{t\to\infty}} \|e(t)\| \leq C_e^+ \Delta, \quad \text{(A3)}$$

where

$$C_e^+ = \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}} \frac{L_\varphi + |K|}{\mu}. \quad \text{(A4)}$$

The inequality (A3) shows that the total synchronization error is proportional to the upper bound on the transmission error Δ.

As was shown in [12], a binary coder is optimal in the sense of the bit-per-second rate, and the optimal sampling time T_s for this coder is

$$T_s = \Delta/(\beta L_y), \quad \text{(A5)}$$

where $\beta \approx 1.688$. Then the channel bit rate $R = 1/T_s$ is as follows:

$$R = \beta L_y/\Delta, \quad \text{(A6)}$$

and this bound is tight for the considered class of coders.

APPENDIX B: HMP PROPERTY AND PASSIFICATION

Consider a linear system

$$\dot{e} = Ae + B\xi(t), \quad \varepsilon = Ce, \quad \text{(B1)}$$

with transfer function $W(\lambda) = C(\lambda I - A)^{-1}B = b(\lambda)/a(\lambda)$, where $b(\lambda)$ and $a(\lambda)$ are polynomials, the degree of $a(\lambda)$ is n, and the degree of $b(\lambda)$ is not greater than $n-1$. The system is called hyperminimum phase, if $b(\lambda)$ is a Hurwitz (stable) polynomial of degree $n-1$ with positive coefficients. To find the existence conditions for a quadratic Lyapunov function required for passification we need the following result.

Passification theorem [22,19]. Consider a linear system with feedback

$$\dot{e} = A_K e + B\xi(t), \quad \varepsilon = Ce, \quad A_K = A - BKC. \qquad \text{(B2)}$$

There exist a positive-definite matrix $P = P^T > 0$ and a number K such that

$$V(e(t)) \leqslant V(e(0)) + \int_0^t \varepsilon(s)^T \xi(s) ds, \qquad \text{(B3)}$$

where $V(e) = e^T Pe$; $e(t)$ is a solution of (B2) if and only if $W(\lambda)$ is a HMP.

In addition, let us show that there exist a quadratic form $V(e) = e^T Pe$ and a number K such that the time derivative $\dot{V}(e)$ of $V(e)$ along trajectories of (B2) satisfies dissipative relation $\dot{V}(e) < 0$ for $\xi\varepsilon \geqslant 0$, $e \neq 0$, if and only if $W(\lambda)$ is HMP. To this end assume that K is fixed. Then the above relation is equivalent to existence of the matrix $P = P^T > 0$ such that $e^T P(A_K e + B\xi) + \xi Ce < 0$ for $e \neq 0$. Since ξ is arbitrary, the latter, in turn, is equivalent to the matrix relations $PA_K + A_K^T P < 0$, $PB = C^T$, and, by the passification theorem, to the HMP condition.

[1] I. I. Blekhman, *Synchronization in Science and Technology* (ASME Press, New York, 1988).
[2] A. Pikovsky, M. Rosenblum, and J. Kurths, *Synchronization: A Universal Concept in Nonlinear Sciences* (Cambridge University Press, Cambridge, U.K., 2001).
[3] L. M. Pecora and T. L. Carroll, Phys. Rev. Lett. **64**, 821 (1990).
[4] I. I. Blekhman, A. L. Fradkov, H. Nijmeijer, and A. Y. Pogromsky, Syst. Control Lett. **31**, 299 (1997).
[5] A. L. Fradkov and A. Y. Pogromsky, *Introduction to Control of Oscillations and Chaos* (World Scientific, Singapore, 1998).
[6] A. L. Fradkov, *Cybernetical Physics: From Control of Chaos to Quantum Control* (Springer-Verlag, Berlin, 2007).
[7] G. N. Nair and R. J. Evans, Automatica **39**, 585 (2003).
[8] G. N. Nair and R. J. Evans, SIAM J. Control Optim. **43**, 413 (2004).
[9] G. N. Nair, R. J. Evans, I. Mareels, and W. Moran, IEEE Trans. Autom. Control **49**, 1585 (2004).
[10] L. M. J. Bazzi and S. K. Mitter, IEEE Trans. Inf. Theory **51**, 2103 (2005).
[11] G. N. Nair, F. Fagnani, S. Zampieri, and R. Evans, Proc. IEEE **95**, 108 (2007).
[12] A. L. Fradkov, B. Andrievsky, and R. J. Evans, Phys. Rev. E **73**, 066209 (2006).
[13] H. Nijmeijer and I. M. Y. Mareels, IEEE Trans. Circuits Syst., I: Fundam. Theory Appl. **44**, 882 (1997).
[14] V. L. Syrmos, C. T. Abdallah, P. Dorato, and K. Grigoriadis, Automatica **33**, 125 (1997).
[15] J. D. Stefanovski and G. M. Dimirovski, Int. J. Syst. Sci. **37**, 643 (2006).
[16] R. W. Brockett and D. Liberzon, IEEE Trans. Autom. Control **45**, 1279 (2000).
[17] D. Liberzon, Automatica **39**, 1543 (2003).
[18] S. Tatikonda and S. Mitter, IEEE Trans. Autom. Control **49**, 1056 (2004).
[19] A. L. Fradkov, I. V. Miroshnik, and V. O. Nikiforov, *Nonlinear and Adaptive Control of Complex Systems* (Kluwer, Dordrecht, 1999).
[20] V. S. Afraimovich, S. N. Chow, and J. K. Hale, Physica D **103**, 442 (1997).
[21] V. N. Belykh, I. V. Belykh, and M. Hasler, Physica D **195**, 159 (2004).
[22] A. L. Fradkov, Autom. Remote Control (Engl. Transl.) **35**, 1960 (1974).